MAE 343. Intermediate Mechanics

Chapter 3: Loads and Stress–Mohr's Circle 3D

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Review Mohr's circle 2D

$2\theta_p$ goes from A to σ_I, always from $A \rightarrow \sigma_I$, and the angle is defined by noting that a positive angle is ccw. In this case:

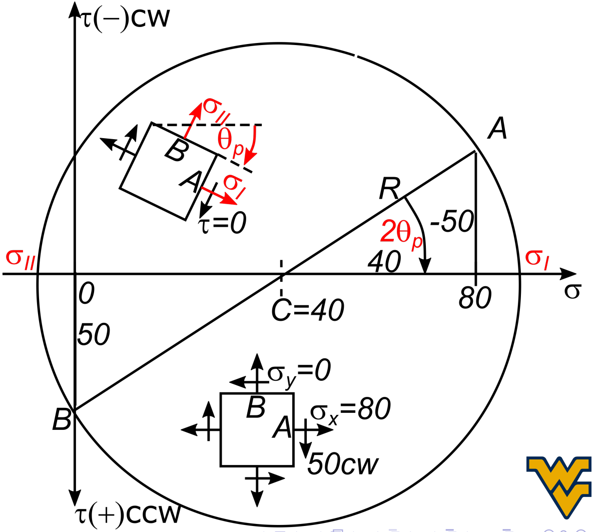
$$C = (0+80)/2 = 40; \quad R = \sqrt{40^2 + 50^2} = 64$$

$$\sigma_I = C + R = 104$$

$$\sigma_{II} = C - R = -24$$

$$\theta_p = -25.67$$

$$\tau_{max} = R, \text{ at the bottom of the diagram}$$



Principal stresses and principal directions

In 2D there are 3 components of stress, shear τ_{xy} and two normal stresses σ_x and σ_y, which can be arranged in a matrix

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

For fixed loads, values of stress change withe the orientation θ of the stress element
For an orientation rotated by θ_p, shear is zero and the two normal stresses are the highest and lowest of all possible.
The definition of principal stresses is that they are the stresses for which shear is zero
The principal stresses are oriented along principal directions θ_p and $\theta_p + \pi/2$
The principal stresses are the principal values of the stress matrix.
The principal directions are the principal directions.
Principal values and directions of a matrix can be obtained by solving this matrix equation

$$\det([\sigma] - \lambda[I]) = 0$$

where $[I]$ is the identity matrix $[I] = [[1, 0]; [0, 1]]$.

Mohr's circle 2D with equations

Casting the graph into equations we get

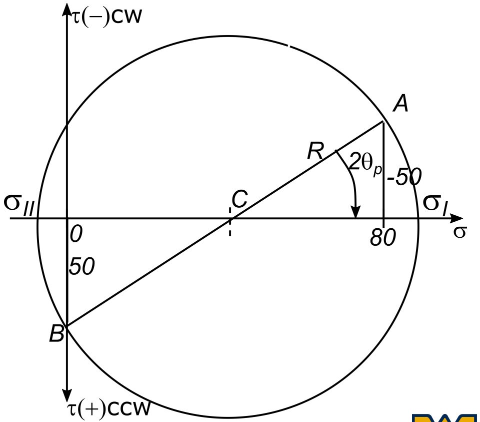
$$\lambda = C \pm R = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

which is simpler, but the same as the eq. in the textbook

$$\lambda = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 - \sigma_x \sigma_y}$$

Then

$$\sigma_I = \lambda_1; \quad \sigma_{II} = \lambda_2$$



Mohr's circle 2D with MATRIX equations

The eq. in the textbook can be derived like this

$$\det\left(\begin{bmatrix}\sigma_x & \tau_{xy}\\ \tau_{xy} & \sigma_y\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0 \rightarrow \det\left(\begin{bmatrix}\sigma_x - \lambda & \tau_{xy}\\ \tau_{xy} & \sigma_y - \lambda\end{bmatrix}\right) = 0$$

$$(\sigma_x - \lambda)(\sigma_y - \lambda) - \tau_{xy}^2 = 0$$

$$\lambda^2 - \lambda(\sigma_x + \sigma_y) - \tau_{xy}^2 + \sigma_x\sigma_y = 0$$

$$\lambda^2 + b\lambda + c = 0$$

$$\lambda = -b/2 \pm \sqrt{(b/2)^2 - c}$$

$$b = -(\sigma_x + \sigma_y); \quad c = -\tau_{xy}^2 + \sigma_x\sigma_y \rightarrow \text{book's eq.}$$

example: $\sigma_x = 80; \quad \sigma_y = 0; \quad \tau_{xy} = -50 \rightarrow b = -80; \quad c = -50^2$

$$\lambda_{1,2} = 80/2 \pm \sqrt{(80/2)^2 + 50^2} = 40 \pm 64 = \{104, -24\}$$

```
// Scilab.org
mode(0);//formatting
b=-80;
c=-2500;
lambda=-b/2+sqrt((b/2)^2-c)
== 104
lambda=-b/2-sqrt((b/2)^2-c)
== -24
```

3D stresses

In 3D there are 6 components of stress, which can be arranged in a matrix

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Mohr's circle 2D with 3D MATRIX equations

$$\det\left(\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \sigma_y & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \sigma_z\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}\sigma_x - \lambda & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \sigma_y - \lambda & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \sigma_z - \lambda\end{bmatrix}\right) = 0$$

Instead, use software calculate the eigenvalues and eigenvectors of

$$\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \sigma_y & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \sigma_z\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = 0$$

For our example

$$\sigma_x = 80; \quad \sigma_y = 0; \quad \tau_{xy} = -50$$
$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

the 2x2 matrix equation becomes

$$\begin{bmatrix}80 & -50 & 0\\ -50 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = 0$$

Solve with Scilab

$$\begin{bmatrix} 80 & -50 & 0 \\ -50 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

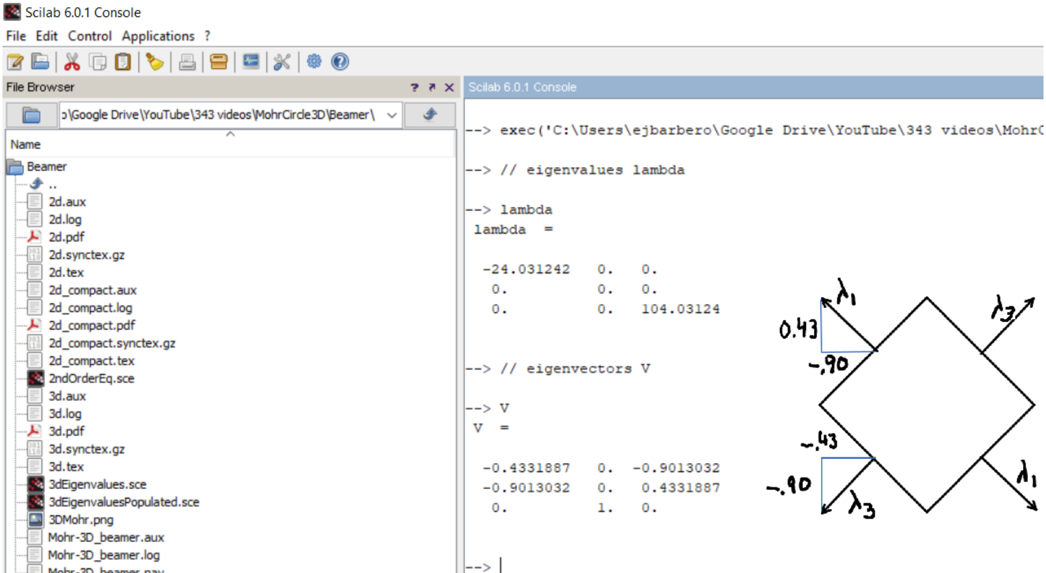
$$[\sigma] - \lambda[I] = 0$$

where σ is the 3x3 matrix of stress, and [I] is the 3x3 identity matrix.

Use function spec() in Scilab, or eig() in MATLAB.

```
// Scilab.org
mode(0);//behave as MATLAB
sigmax=80;
sigmay=0;
sigmaz=0;
tauxy=-50;
tauxz=0;
tauyz=0;
sigma=[[sigmax,tauxy,tauxz];
      [tauxy,sigmay,tauyz];
      [tauxz,tauyz,sigmaz]];
[V,lambda]=spec(sigma);//do not print
lambda
== 104
== -24
== 0
```

2D Eigenvectors, Eigenvalues, manual selection $\lambda_1 > \lambda_2 > \lambda_3$



Mohr's circle 3D, $\sigma_z \neq 0$

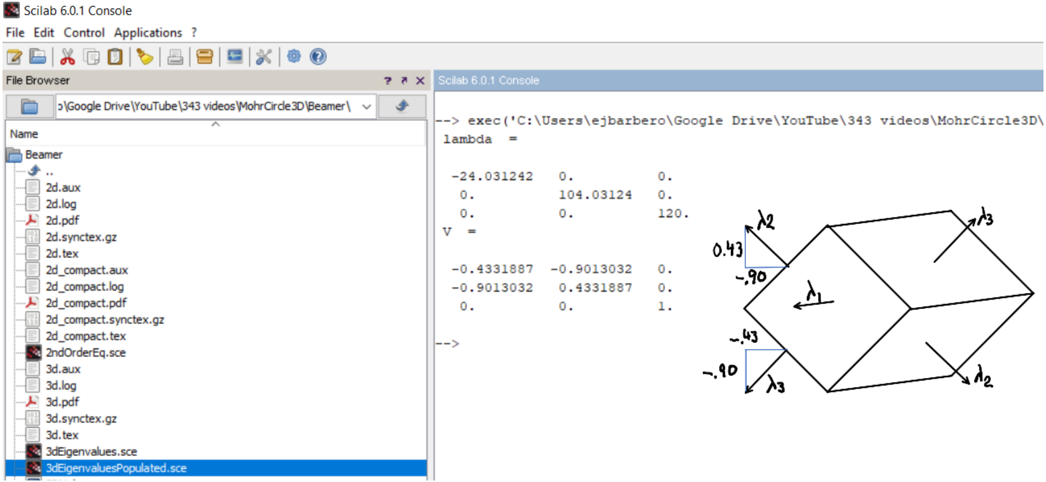
- From the previous slide, we have the 2D Mohr's circle by software, done!
- If the 3x3 stress matrix is fully populated

$$\begin{bmatrix} 80 & -50 & 0 \\ -50 & 0 & 0 \\ 0 & 0 & 120 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

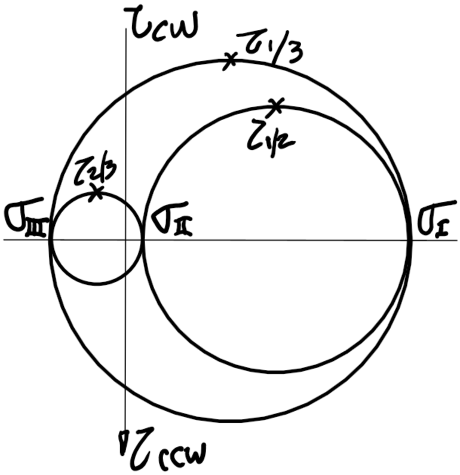
$$[\sigma] - \lambda[I] = 0$$

```
// Scilab.org
mode(0);//behave as MATLAB
sigmax=80;
sigmay=0;
sigmaz=120;//notice this
tauxy=-50;
tauxz=0;
tauyz=0;
sigma=[[sigmax,tauxy,tauxz];
    [tauxy,sigmay,tauyz];
    [tauxz,tauyz,sigmaz]];
[V,lambda]=spec(sigma,I)
== 104
== -24
== 120//notice this
```

3D Eigenvectors, Eigenvalues, manual selection $\lambda_1 > \lambda_2 > \lambda_3$



Now we draw Mohr's circle 3D



- We cannot draw it until we calculate the 3 principal stresses
- Must order the values so that $\sigma_I > \sigma_{II} > \sigma_{III}$
- Notice that τ_{max} is $\tau_{1/3} = (\sigma_I - \sigma_{III})/2$, not $\tau_{1/2}$
- So, we need all 3 stress values to calculate τ_{max}

Summary

- We cannot draw the 3D Mohr's circle until we calculate the 3 principal stresses
- Must order the values so that $\sigma_I > \sigma_{II} > \sigma_{III}$
- We need all 3 principal stress values to calculate $\tau_{max} = (\sigma_1 - \sigma_3)/2$
- Why τ_{max}? because it controls yield, plasticity, and yield safety factor.
- The eigenvectors are used to predict the *directions* of failure, as we shall see in chapter 5.